

# UNCOUNTABLY MANY NON-COMMENSURABLE FINITELY PRESENTED PRO- $p$ GROUPS

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ABSTRACT. Let  $m \geq 3$  be a positive integer. We prove that there are uncountably many non-commensurable metabelian uniform pro- $p$  groups of dimension  $m$ . Consequently, there are uncountably many non-commensurable finitely presented pro- $p$  groups with minimal number of generators  $m$  (and minimal number of relations  $\binom{m}{2}$ ).

## 1. INTRODUCTION

Throughout let  $p$  be a prime. In [3], Lazard gave a comprehensive treatment of the theory of  $p$ -adic analytic groups. Later Lubotzky and Mann re-interpreted the group-theoretic aspects of Lazard's work by introducing the concept of powerful pro- $p$  group (see [4]). A pro- $p$  group  $G$  is said to be *powerful* if  $p \geq 3$  and  $[G, G] \leq G^p$ , or  $p = 2$  and  $[G, G] \leq G^4$ . Here,  $[G, G]$  and  $G^p$  denote the (closures of the) commutator subgroup and the subgroup generated by all  $p$ th powers. Using this terminology, Lazard's main result is the following algebraic characterisation of  $p$ -adic analytic groups: a topological group is  $p$ -adic analytic if and only if it contains an open subgroup which is a finitely generated powerful pro- $p$  group.

A powerful pro- $p$  group  $G$  is called *uniform* if it is finitely generated and torsion-free. A key feature of a uniform pro- $p$  group  $G$  is that its minimal number of (topological) generators  $d(G)$  coincides with the dimension of  $G$  as a  $p$ -adic manifold.

In this short note we prove the following.

**Theorem 1.1.** *Let  $m \geq 3$  be a positive integer. There are uncountably many non-commensurable metabelian uniform pro- $p$  groups of dimension  $m$ . Consequently, there are uncountably many non-commensurable finitely presented pro- $p$  groups with minimal number of generators  $m$  (and minimal number of relations  $\binom{m}{2}$ ).*

As a direct consequence we get the following.

**Corollary 1.2.** *Let  $m \geq 3$  be a positive integer. There are uncountably many non-commensurable finitely presented pro- $p$  groups  $G$  with  $d(G) = m$  which are not a pro- $p$  completion of any finitely presented abstract group. In particular, these*

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groups do not have a finite presentation with all the relations of finite length (as words on the generators).

Let  $\Gamma$  be a finitely generated abstract group or a finitely generated profinite group, and for  $n \in \mathbb{N}$ , let  $a_n(\Gamma)$  denote the number of subgroups of index  $n$  in  $\Gamma$ . The zeta function of  $\Gamma$  is given by the Dirichlet series

$$\zeta_\Gamma(s) = \sum_{n=1}^{\infty} a_n(\Gamma) n^{-s}.$$

Similarly, the normal zeta function of  $\Gamma$  is defined by

$$\zeta_\Gamma^\triangleleft(s) = \sum_{n=1}^{\infty} a_n^\triangleleft(\Gamma) n^{-s},$$

where  $a_n^\triangleleft(\Gamma)$  denotes the number of normal subgroups of index  $n$  in  $\Gamma$ .

We say that two groups  $H$  and  $K$  are *isospectral* (*normally isospectral*) if their zeta functions (normal zeta functions) are the same. In [5] is given an example of an infinite family of non-commensurable normally isospectral pro- $p$  groups.

Du Sautoy has proved that if  $G$  is a compact  $p$ -adic analytic group then  $\zeta_G(s)$  and  $\zeta_G^\triangleleft(s)$  are rational functions over  $\mathbb{Q}$  of  $p^{-s}$  (see [2]). This result together with Theorem 1.1 imply the following.

**Corollary 1.3.** *There are uncountably many non-commensurable isospectral (normally isospectral) pro- $p$  groups.*

The question of whether there are uncountably many non-isomorphic finitely presented pro- $p$  groups was recently explicitly raised by A. Lubotzky at the conference “Geometric and Combinatorial Group Theory” in honor of E. Rips and by E. Zelmanov at the conference “XX Coloquio Latinoamericano de Álgebra” (Zelmanov attributed the question to Lubotzky). I am grateful to E. Zelmanov and R. Grigorchuk for encouraging me to write down this paper.

## 2. POWERFUL PRO- $p$ GROUPS AND LIE ALGEBRAS

Let  $G$  be a pro- $p$  group. The lower central  $p$ -series of  $G$  is defined as follows:  $P_1(G) = G$  and  $P_{i+1}(G) = P_i(G)^p [P_i(G), G]$  for  $i \in \mathbb{N}$ . One can use this series to define uniform pro- $p$  groups. Indeed, a pro- $p$  group is uniform if and only if it is finitely generated, powerful and  $|P_i(G) : P_{i+1}(G)| = |G : P_2(G)|$  for all  $i \in \mathbb{N}$ . This definition of a uniform pro- $p$  group is equivalent to the definition given in the introduction (see [1, Theorem 4.5]).

Given a powerful pro- $p$  group  $G$  and  $n \in \mathbb{N}$ , we have  $P_{n+1}(G) = G^{p^n} = \{x^{p^n} \mid x \in G\}$  (see [1, Theorem 3.6]). Moreover, if  $G$  is uniform, then the mapping  $x \mapsto x^{p^n}$  is a homeomorphism from  $G$  onto  $G^{p^n}$  (see [1, Lemma 4.10]). This shows that each element  $x \in G^{p^n}$  admits a unique  $p^n$ th root in  $G$ , which we denote by  $x^{p^{-n}}$ .

Analogous to the case of pro- $p$  groups, a  $\mathbb{Z}_p$ -Lie algebra  $L$  is called *powerful* if  $L \cong \mathbb{Z}_p^d$  for some  $d > 0$  as  $\mathbb{Z}_p$ -module and  $(L, L)_{\text{Lie}} \subseteq pL$  ( $(L, L)_{\text{Lie}} \subseteq 4L$  if  $p = 2$ ).

If  $G$  is an analytic pro- $p$  group, then it has a characteristic open subgroup which is uniform. For every open uniform subgroup  $H \leq G$ ,  $\mathbb{Q}_p[H]$  can be made into a normed  $\mathbb{Q}_p$ -algebra, call it  $A$ , and  $\log(H)$ , considered as a subset of the completion  $\hat{A}$  of  $A$ , will have the structure of a Lie algebra over  $\mathbb{Z}_p$ . There is a different construction of an intrinsic Lie algebra over  $\mathbb{Z}_p$  for uniform groups. The uniform group  $U$  and its Lie algebra over  $\mathbb{Z}_p$ , call it  $L_U$ , are identified as sets, and the Lie operations are defined by

$$g+h = \lim_{n \rightarrow \infty} (g^{p^n} h^{p^n})^{p^{-n}}, \quad (g, h)_{Lie} = \lim_{n \rightarrow \infty} [g^{p^n}, h^{p^n}]^{p^{-2n}} = \lim_{n \rightarrow \infty} (g^{-p^n} h^{-p^n} g^{p^n} h^{p^n})^{p^{-2n}}.$$

It turns out that  $L_U$  is a powerful  $\mathbb{Z}_p$ -Lie algebra and it is isomorphic to the  $\mathbb{Z}_p$ -Lie algebra  $\log(U)$ . On the other hand, if  $L$  is a powerful  $\mathbb{Z}_p$ -Lie algebra, then the Campbell-Hausdorff formula induces a group structure on  $L$ ; the resulting group is a uniform pro- $p$  group. If this construction is applied to the  $\mathbb{Z}_p$ -Lie algebra  $L_U$  associated to a uniform group  $U$ , one recovers the original group. Indeed, the assignment  $U \mapsto L_U$  gives an isomorphism between the category of uniform pro- $p$  groups and the category of powerful  $\mathbb{Z}_p$ -Lie algebras (see [1, Theorem 9.10]). A detailed treatment of  $p$ -adic analytic groups is given in [1].

### 3. PROOF OF THEOREM 1.1

Let us denote by  $\mathbb{Z}_p^*$  the group of units of the  $p$ -adic integers, i.e.,  $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$ .

**Proposition 3.1.** *Let  $d \in \mathbb{Z}_p^*$  and for  $n \geq 1$ , let  $L_{2n+1}(d)$  and  $L_{2n+2}(d)$  be  $\mathbb{Z}_p$ -Lie algebras defined in the following way.*

- (i)  $L_{2n+1}(d)$  is the free  $\mathbb{Z}_p$ -module on the basis  $\{x, e_2, \dots, e_{2n+1}\}$  and the Lie bracket is given as follows:

$$[e_i, e_j] = 0 \text{ for } 2 \leq i, j \leq 2n+1, \quad [e_2, x] = de_{2n+1},$$

$$[e_i, x] = e_{2n-i+3} \text{ for } 3 \leq i \leq 2n \text{ and } [e_{2n+1}, x] = e_2 + e_{2n+1}.$$

- (ii)  $L_{2n+2}(d)$  is the free  $\mathbb{Z}_p$ -module on the basis  $\{x, e_2, \dots, e_{2n+2}\}$  and the Lie bracket is given as follows:

$$[e_i, e_j] = 0 \text{ for } 2 \leq i, j \leq 2n+2, \quad [e_2, x] = de_{2n+2},$$

$$\text{and } [e_i, x] = e_{2n-i+4} \text{ for } 3 \leq i \leq 2n+2.$$

Then  $L_k(d) \cong L_k(l)$  if and only if  $d = l$ . Moreover,  $d$  is an invariant of the isomorphism type of the  $\mathbb{Q}_p$ -Lie algebra  $L_k(d) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ .

*Proof.* Let  $k \geq 3$  and let  $L = L_k(d)$ . It is easy to see that  $L$  is a well-defined  $\mathbb{Z}_p$ -Lie algebra and that  $L' = [L, L]$  is an abelian ideal generated by  $e_2, e_3, \dots, e_k$ . Let  $A(x)$  denote the restriction of  $\text{ad} x$  to  $L'$  and let  $A_k(d)$  be the matrix associated

to this linear transformation with respect to the basis  $e_2, \dots, e_k$ . If  $k = 2n + 1$  we have the following  $2n \times 2n$  matrix

$$A_{2n+1}(d) = \begin{pmatrix} 0 & \cdots & 0 & 0 & d \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and if  $k = 2n + 2$  we have the following  $(2n + 1) \times (2n + 1)$  matrix

$$A_{2n+2}(d) = \begin{pmatrix} 0 & \cdots & 0 & 0 & d \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that  $\text{tr}(A_k(d)) = 1$  for all  $k \geq 3$ . Moreover,  $\det(A_{2n+1}(d)) = (-1)^n d$  and  $\det(A_{2n+2}(d)) = (-1)^n d$  for all  $n \geq 1$  (i.e.,  $\det(A_k(d)) = (-1)^{\lfloor \frac{k-1}{2} \rfloor} d$  for  $k \geq 3$ ).

Also note that  $L = L' \oplus x\mathbb{Z}_p$ . Moreover, if also  $L = L' \oplus y\mathbb{Z}_p$  for some  $y \in L$ , then  $y = ux + e$  with  $u \in \mathbb{Z}_p^*$  and  $e \in L'$ , and

$$[e_i, y] = [e_i, ux + e] = u[e_i, x] + [e_i, e] = u[e_i, x].$$

Thus  $A(y) = \text{ad}_{y|_{L'}} = uA(x)$ , so  $\text{tr}A(y) = u\text{tr}A(x)$  and  $\det A(y) = u^{k-1}\det A(x)$ , where  $k - 1 = \dim L'$ . Since  $\text{tr}A(x) = \text{tr}(A_k(d)) = 1$ , it follows that

$$(\text{tr}A(y))^{-(k-1)} \cdot \det A(y) = u^{1-k} \cdot \det A(y) = \det A(x) = (-1)^{\lfloor \frac{k-1}{2} \rfloor} d$$

is an invariant of  $L = L_k(d)$ . Thus  $L_k(d) \cong L_k(l)$  if and only if  $d = l$ .

Finally, note that our proof works equally well with  $\mathbb{Q}_p$  in place of  $\mathbb{Z}_p$ . Thus  $d$  is an invariant of the isomorphism type of the  $\mathbb{Q}_p$ -Lie algebra  $L_k(d) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ .  $\square$

**Corollary 3.2.** *Let  $k \geq 3$  and let  $d, l \in \mathbb{Z}_p^*$ . The  $\mathbb{Z}_p$ -Lie algebra  $p^2 L_k(d)$  is powerful and  $p^2 L_k(d) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong p^2 L_k(l) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  if and only if  $d = l$ . In particular, there are uncountably many non-isomorphic (powerful)  $\mathbb{Z}_p$ -Lie algebras of rank  $k$ .*

*Proof.* We have  $[p^2 L_k(d), p^2 L_k(d)] = p^4 [L_k(d), L_k(d)] \subseteq p^2(p^2 L_k(d))$ . Hence,  $p^2 L_k(d)$  is a powerful  $\mathbb{Z}_p$ -Lie algebra. Now note that  $f_1, \dots, f_k$  is a basis of  $L_k(d) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  if and only if  $p^2 f_1, \dots, p^2 f_k$  is a basis of  $p^2 L_k(d) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ . Hence,  $p^2 L_k(d) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong p^2 L_k(l) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  if and only if  $L_k(d) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong L_k(l) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ , and by Proposition 3.1, if and only if  $d = l$ . The last part of the corollary follows directly from the fact that  $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$  is an uncountable set.  $\square$

We need the following well-known proposition.

**Proposition 3.3** ([1, Proposition 4.32]). *Let  $G$  be a uniform pro- $p$  group of dimension  $d = d(G)$  and let  $X = \{x_1, \dots, x_d\}$  be a topological generating set for  $G$ . Then  $G$  has a presentation  $\langle X; R \rangle$ , where*

$$R = \{[x_i, x_j]x_1^{a_1(i,j)} \cdots x_d^{a_d(i,j)} \mid 1 \leq i, j \leq d\},$$

*and for each  $m, i$  and  $j$ ,  $a_m(i, j) \in p\mathbb{Z}_p$  if  $p$  is odd,  $a_m(i, j) \in 4\mathbb{Z}_2$  if  $p = 2$ . In particular,  $G$  is finitely presented.*

*Proof of Theorem 1.1.* Let  $m \geq 3$  be a positive integer. By Corollary 3.2, we know that  $p^2L_m(d)$  is a powerful  $\mathbb{Z}_p$ -Lie algebra of rank  $m$  for all  $d \in \mathbb{Z}_p^*$ . By [1, Theorem 9.8], we can associate to  $p^2L_m(d)$  a uniform pro- $p$  group  $G_m(d)$  which has the same underlying set as  $p^2L_m(d)$  and such that  $d(G_m(d)) = m$ . By [1, Theorem 9.10] it follows that  $G_m(d) \cong G_m(l)$  if and only if  $p^2L_m(d) \cong p^2L_m(l)$ . Moreover,  $G_m(d)$  and  $G_m(l)$  are pairwise non-commensurable whenever  $d \neq l$ , since  $p^2L_k(d) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong p^2L_k(l) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  if and only if  $d = l$ .

Now by Corollary 3.2, we have uncountably many uniform pro- $p$  groups  $G$  such that  $d(G) = m$  and, by Proposition 3.3, all these groups are finitely presented. Moreover, by [1, Theorem 4.35], the minimal number of relations of  $G_m(d)$  is  $\binom{m}{2}$ .  $\square$

*Remark 3.4.* Note that one can give an explicit presentation for  $G_m(d)$ . For example, the uniform pro- $p$  group associated to the powerful  $\mathbb{Z}_p$ -Lie algebra  $p^2L_4(d)$  is given by the presentation

$$\begin{aligned} G_4(d) = \langle y, z_2, z_3, z_4 \mid & [z_2, z_3] = 1, [z_3, z_4] = 1, \\ & [z_4, z_2] = 1, [z_2, y] = z_4^{dp^2}, [z_3, y] = z_3^{p^2}, [z_4, y] = z_2^{p^2} \rangle. \end{aligned}$$

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